Opportunities for Automatic Differentiation in Stochastic Differential Equation Solving

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Overview

- SDEs and numerical methods
- Current software
- Three opportunities for AD

"**It may very well be said that the best way to understand SDEs is to work with their numerical solutions.**"

Salih N. Neftci, in An Introduction to the Mathematics of Financial Derivatives, Academic Press, 2nd Edition, 2000.

Where do SDEs arise?

“**Rigorous**”

- Particle models in physics
- Chemical kinetics with intrinsic noise

“**Phenomenological**”

- Finance
- Population dynamics
- Mechanics
- Any model that needs fixing .....

Euler–Maruyama for a scalar (Ito) SDE

Given functions f and g, with timestep Δt:

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t \, f(\mathbf{X}_n) + \Delta \mathbf{W}_n \, g(\mathbf{X}_n)$$

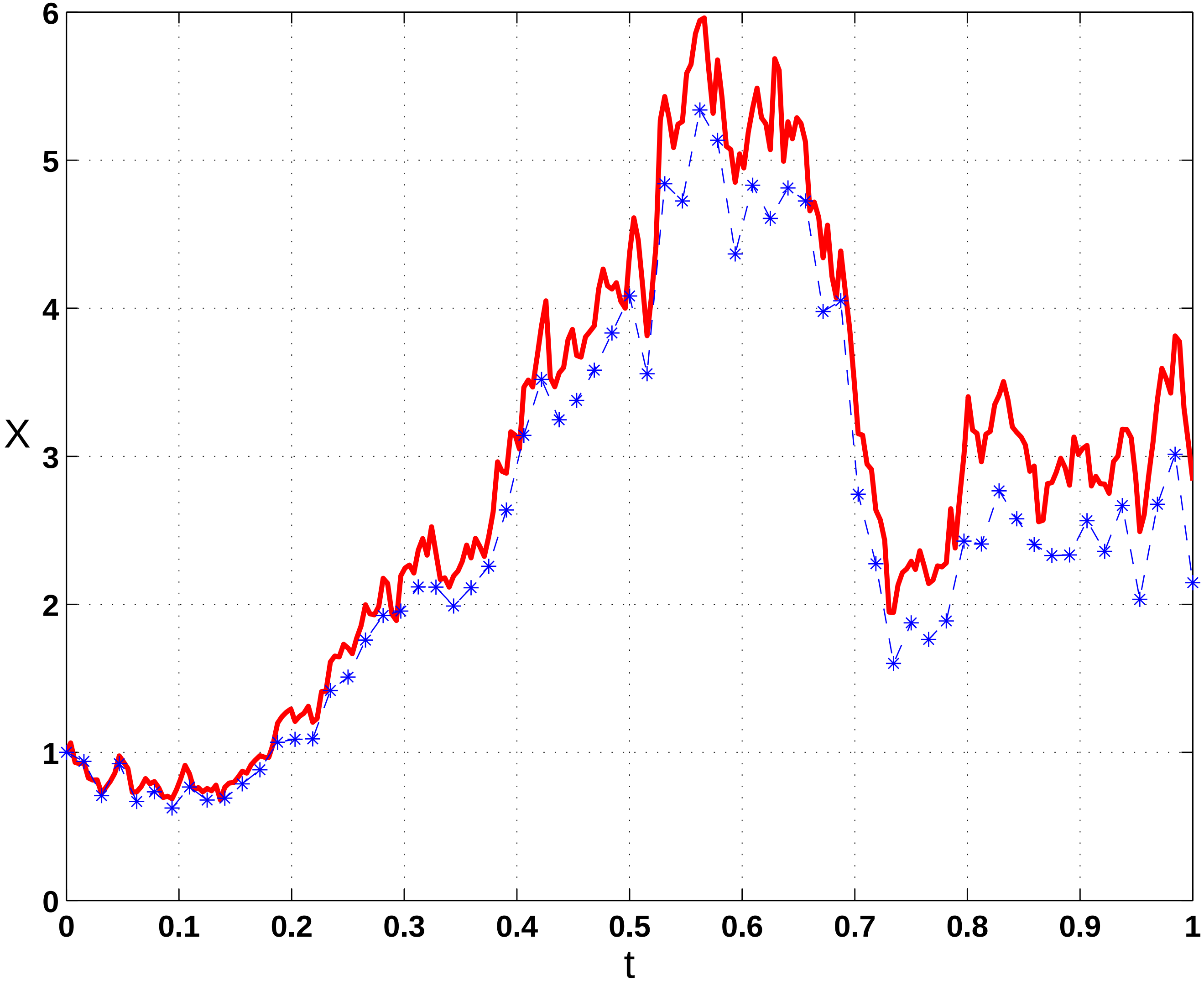
where $\Delta \mathbf{W}_n \sim N(0, \Delta t)$ are independent

In MATLAB, $\Delta \mathbf{W}_n$ becomes `sqrt(Dt)*randn`

Limit $\Delta t \to 0$ defines an SDE

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t)$$

Linear Case: $f(x) = \mu x$ and $g(x) = \sigma x$



$\mathbf{X}_n \approx \mathbf{X}(t_n)$: what about convergence?

Euler–Maruyama has **weak order** 1:

$$\sup_{0 \leq t_n \leq T} \left| \mathbb{E}\left[\Phi(\mathbf{X}_n)\right] - \mathbb{E}\left[\Phi(\mathbf{X}(t_n))\right] \right| \leq K \Delta t$$

and **strong order** $\frac{1}{2}$:

$$\sup_{0 \leq t_n \leq T} \mathbb{E}\left[\left| \mathbf{X}_n - \mathbf{X}(t_n) \right| \right] \leq K \Delta t^{\frac{1}{2}}$$

Milstein's method

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t \, f(\mathbf{X}_n) + \Delta \mathbf{W}_n \, g(\mathbf{X}_n) + \tfrac{1}{2} g(\mathbf{X}_n) g'(\mathbf{X}_n) \left(\Delta \mathbf{W}_n^2 - \Delta t \right)$$

has **strong order** 1

Relevant Issues

- **Implicit methods** have better stability:

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t\, f(\mathbf{X}_{n+1}) + \Delta \mathbf{W}_n\, g(\mathbf{X}_n)$$

- **Higher order methods** are very expensive in general
- But, many SDEs have **special structure** (small noise, additive noise, commutative noise, symplectic) that allows customized high order methods to be derived

Current Software

MATLAB

Hagen Gilsing & Tony Shardlow, *SDELab - stochastic differential equations with MATLAB*, Technical Report, August 2005, University of Manchester,
http://www.maths.man.ac.uk/~shardlow/publications.html

- “Better” random number generation
- Efficient treatment of multiple stochastic integrals
- Evaluation of $g'(\mathbf{X}_n)$ in Milstein by user-supplied function or finite difference accurate to $O(\sqrt{\Delta t})$

MAPLE

Sasha Cyganowski, *The MAPLE Stochastic Package*,
http://www.math.uni-frankfurt.de/~numerik/maplestoch/

- Uses MAPLE to derive high/order customized methods

Opportunities for AD: Part I

Milstein's Method

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t\, f(\mathbf{X}_n) + \Delta \mathbf{W}_n\, g(\mathbf{X}_n) + \tfrac{1}{2} g(\mathbf{X}_n) g'(\mathbf{X}_n) \left(\Delta \mathbf{W}_n^2 - \Delta t \right)$$

Evaluate $g'(\mathbf{X}_n)$ using AD rather than finite difference

For systems of SDEs this becomes a Jacobian matrix evaluation

Opportunities for AD: Part II

Implicit Methods such as

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t\, f(\mathbf{X}_{n+1}) + \Delta \mathbf{W}_n\, g(\mathbf{X}_n)$$

"**Frozen Jacobian**" trick from ODEs is less useful:

- solution is never smoothly varying
- change from step to step is $O(\sqrt{\Delta t})$
- variable stepsize is not so easy

Opportunities for AD: Part III

Customized Methods involving derivatives of f and g can be derived symbolically. Clear potential for AD.

$\Rightarrow$ floating point version of *MAPLE Stochastic Package*?

E.g. **Weak order 2 scheme**:

$$\begin{aligned}
\mathbf{X}_{n+1} = \mathbf{X}_n &+ \Delta t\, f(\mathbf{X}_n) + \Delta \mathbf{W}_n\, g(\mathbf{X}_n) \\
&+ \tfrac{1}{2}\left(f(\mathbf{X}_n) f'(\mathbf{X}_n) + \tfrac{1}{2} g(\mathbf{X}_n)^2 f''(\mathbf{X}_n)\right) \Delta t^2 \\
&+ \left(f(\mathbf{X}_n) g'(\mathbf{X}_n) + \tfrac{1}{2} g(\mathbf{X}_n)^2 g''(\mathbf{X}_n)\right) \left(\Delta t \Delta \mathbf{W}_n - \Delta \mathbf{Z}_n\right) \\
&+ g(\mathbf{X}_n) f'(\mathbf{X}_n) \Delta \mathbf{Z}_n \\
&+ \tfrac{1}{2} g(\mathbf{X}_n) g'(\mathbf{X}_n) \left(\Delta \mathbf{W}_n^2 - \Delta t\right)
\end{aligned}$$

Some accessible references

Theory:

- *Basic Stochastic Processes*,
  Z. Brzezniak and T. Zastawniak, Springer, 1999
- *Stochastic Differential Equations and Applications*,
  X. Mao, Horwood, 1997
- *Introduction to Stochastic Calculus with Applications*,
  2nd Ed., F. Klebaner, Imp. Coll. Press, 2005
- *Elementary Stochastic Calculus with Finance in View*,
  T. Mikosch, World Scientific, 1998

Numerics:

- *Numerical Solution of Stochastic Differential Equations*,
  P. Kloeden and E. Platen, Springer, 3rd printing 1999
- *An algorithmic introduction to numerical simulation of stochastic differential equations*, D. Higham, SIAM Review, Education Section, 2001